Q21 Given S be a finite set of positive integers.

Assume that there are precisely 2023 ordered pairs (x,y) in $S \times S$ such that xy is a perfect square.

for example

$S = \{1, 2, 4\}$

$\Rightarrow S \times S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)\}$

$\Rightarrow$ in $S \times S$

$\{(1,1), (1,4), (2,2), (4,1), (4,4)\}$

$\Rightarrow$ are exactly 5 pairs with xy is a perfect square and 4 are non-perfect square.

let $|S| = n$

then $|S \times S| = n^2$

$\Rightarrow n^2 > 2023$

$\Rightarrow$ minimum $n > 45$

to get 2023 ordered pairs to get perfect squares of xy.

and $2023 = 7 \times 17^2$

let S has n elements in that k are non perfect squares and $(n-k)$ are perfect squares.

so perfect squares with perfect squares always give a pair which is perfect square and (non-perfect) with it self is a perfect square

ie- $(n-k, n-k)$ is always a perfect squares and (k, k) is always perfect squares.

$\Rightarrow$ According to questions

$$(n-k)^2 + k^2 = 2023$$

which leads k should be greater than 4.