Indian National Mathematical Olympiad (INMO-2023)

Date: 15-01-2023

Time: 4 Hr QP & SOLUTIONS Max. Marks: 102

01.

Let S be a finite set of positive integers. Assume that there are precisely 2023 ordered pairs (x, y) in $S \times S$ so that the product xy is a perfect square. Prove that one can find at least four distinct elements in S so that none of their pairwise products is a perfect square.

Note: As an example, if $S = \{1, 2, 4\}$, there are exactly five such ordered pairs: $(1, 1)$, $(1, 4)$, $(2, 2)$, $(4, 1)$, and $(4, 4)$.

01. Solution

Given S be a finite set of positive integers.

Assume that there are precisely 2023 ordered pairs (x, y) in $S \times S$ such that xy is a perfect square.

For example

$S = \{1, 2, 4\}$

$\Rightarrow S \times S = \{(1,1),(1,2)(1,3),(2,1)(2,2)(2,4)(4,1)(4,2)(4,4)\}$

$\Rightarrow$ in $S \times S$

$\{(1,1),(1,4),(2,2),(4,1),(4,4)\}$

$\Rightarrow$ are exactly 5 pairs with xy is a perfect square and 4 are non-perfect square.

Let $|S| = n$

Then $|S \times S| = n^2$

$\Rightarrow n^2 > 2023$

Let $(x, y) \in S \times S$

Any xy is a perfect square

$\Rightarrow \quad x = P_1^{\alpha_1}.P_2^{\alpha_2}........P_k^{\alpha_k}$

$y = P_1^{\beta_1}.P_2^{\beta_2}........P_k^{\beta_k}$

Now $xy = P_1^{\alpha_1+\beta_1}.P_2^{\alpha_2+\beta_2}......P_k^{\alpha_k+\beta_k}$

And xy is a perfect square if $\alpha_i + \beta_i$ is even for all $i = 1, 2, ..., k$

Now if $x = y$ then always true.

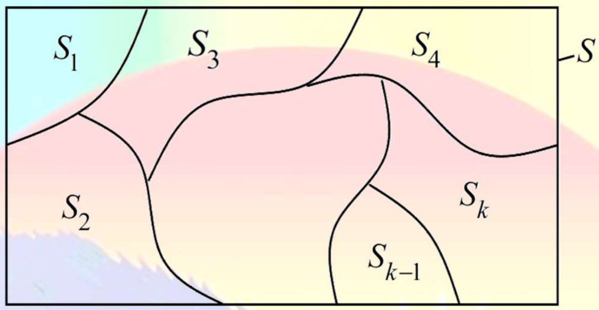
Let k elements are there s.t $x = y$ then $x.y$ is a perfect square.

In another case $x \neq y$ but xy is a perfect square

$\Rightarrow$ If x, y are l numbers then (x, y) are l^2 pairs

$\Rightarrow l^2$ pairs of $S \times S$

Now



$\Rightarrow \boxed{S = \bigcup_{i=1}^{k} S_i}$ and $S_i \cap S_j - \phi$ for $i \neq j$

Proof by contradiction:

Let us assume that it is true for at most three elements

$\Rightarrow S_1 \cup S_2 \cup S_3 = S$ and $|S_1| = k_1$, $|S_2| = k_2$, and $|S_3| = k_3$

Case-1:

If $k_1^2 + k_2^2 + k_3^2 = 2023$

Which has no solution if solve under mod 4 and mod 5

Which is as non-linear Diophantine equation and in same way

Case-2:

If $k_1^2 + k_2^2 = 2023$ also not possible under $(\text{mod } 4)$

Case-3:

If $k_1^2 = 2023$

Which is not at possible

Which is contradiction to our assumption. Hence $k \geq 4$.

02.

Suppose $a_0, \ldots, a_{100}$ are positive reals. Consider the following polynomial for each k in $\{0, 1, \ldots, 100\}$:

$$a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k,$$

where indices are taken modulo 101, *i.e.*, $a_{100+i} = a_{i-1}$ for any i in $\{1, 2, \ldots, 100\}$. Show that it is impossible that each of these 101 polynomials has all its roots real.

02. Solution

$a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} = \ldots\ldots + a_{1+k}x + a_k$

Given

$a_0, a_1, a_2, \ldots.. a_{100} \in \mathbb{R}^+$

$k \in \{0, 1, 2, \ldots.., 100\}$

and $a_{100+i} \equiv a_{i-1}$ mod (101)

our $a_i's$ are solved under mod101

Now put $k = 0, 1, 2, 3, \ldots.., 100$ then

We will get (Consider polynomial Equation)

$a_{100}x^{100} + 100a_{99}x^{99} + a_{98}x^{98} \ldots\ldots a_1x + a_0 = 0$

$a_{101}x^{100} + 100a_{100}x^{99} + a_{99}x^{98} \ldots\ldots a_2x + a_1 = 0$

$a_{102}x^{100} + 100a_{101}x^{99} + a_{100}x^{98} \ldots\ldots a_3x + a_2 = 0$

$a_{200}x^{100} + 100a_{199}x^{99} + a_{198}x^{98} \ldots\ldots a_{101} + a_{100} = 0$

add all the 101 equations and we will apply mod101

and taking out $(a_0 + a_1 + a_2 + \ldots\ldots + a_{100})$

$(a_0 + a_1 + a_2 + \ldots\ldots + a_{100})(x^{100} + 100x^{99} + x^{98} + \ldots.. + x + 1) = 0$

We know that $a_i's \in \mathbb{R}^+$

$\Rightarrow \sum_{i=0}^{100} a_i > 0 \neq 0$

$\Rightarrow x^{100} + 100x^{99} + x^{98} + \ldots.. + x + 1 = 0$ ……….*

By the Descart's Rule of sign this equation has all negative real roots.

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_{100}$ are the roots

$\Rightarrow \alpha_i \in \mathbb{R}^-$

Then by Vieta's Relation

$$\sum_{i=1}^{100} \alpha_i = -100$$

$$\sum_{i=1}^{99} \alpha_i = -1 \Rightarrow S_{99} = -1 \_\_\_\_\_\_\_\_\_\_\_ (1)$$

And $\alpha_1\alpha_2\alpha_3.........\alpha_{100} = -1 \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)$

$\Rightarrow$ Dividing equations (1) and (2)

$$\sum \frac{1}{\alpha_i} = 1$$

Using $AM \geq GM$

$$\sum a_i \geq \sum \frac{1}{a_i}$$

$\Rightarrow 100 \geq 100^2$

All roots are real and equal and the sum of square of roots is -1

Which is not possible

$\Rightarrow$ it is impossible that each of these 101 polynomials has all its roots are real.

03.

Let $\mathbb{N}$ denote the set of all positive integers. Find all real numbers c for which there exists a function $f : \mathbb{N} \to \mathbb{N}$ satisfying:

(a) for any $x, a \in \mathbb{N}$, the quantity $\frac{f(x+a)-f(x)}{a}$ is an integer if and only if $a = 1$;

(b) for all $x \in \mathbb{N}$, we have $|f(x) - cx| < 2023$.

03. Solution

$\mathbb{N} \to$ Natural Number

$f : \mathbb{N} \to \mathbb{N}$

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

$a_0 \in \mathbb{N}$

Then $f(x)$ is an integral polynomial

$$\Rightarrow \frac{f(m) - f(n)}{m - n} \in \mathbb{N}$$

$\Rightarrow f(x+a) \in \mathbb{N}$

And $f(x) \in \mathbb{N}$

$\Rightarrow (x+a) - x \mid f(x+a) - f(x)$

$\Rightarrow a \mid f(x+a) - f(x)$

$\forall x, a \in \mathbb{N}$

By the divisibility property of polynomials $a \mid f(x+a)-f(x)$

Now assume that $f(x)$ is not a polynomial and it is true for

$\frac{f(x+2)-f(x)}{2} \notin \mathbb{Z}$ iff $a=1$

Now let $a=2$

$\Rightarrow \frac{f(x+2)-f(x)}{2} \notin \mathbb{Z}$

$\Rightarrow 2 \nmid f(x+2)-f(x)$

$\Rightarrow f(x+2)-f(x)$ is an odd number

Replace $x \to (x+2)$

$\Rightarrow \frac{f(x+1)-f(x+2)}{2} \notin \mathbb{Z}$

$\Rightarrow 2 \nmid f(x+1)-f(x+2)$ is an odd number

$\Rightarrow \{f(x+4)-f(x+2)\}+\{f(x+2)-f(x)\}$ is an even number

$\Rightarrow f(x+4)-f(x)$ is even

$= 2 \mid f(x+4)-f(x)$

If implies we can say that

$2^2 \mid f(x+2^2)-f(x)$

$2^2 \mid f(x+2^3)-f(x)$

$2^n \mid f(x+2^{n+1})-f(x)$

$\Rightarrow \frac{f(x+2^n)-f(x)}{2^{n-1}} \in \mathbb{Z}$

From second statement

$|f(x)-cx| < 2023$

$\Rightarrow -2023 < f(x)-cx < 2023$

$\Rightarrow cx-2023 < f(x) < cx+2023$

$\Rightarrow c2^k - 4046 \le f(x+2^n)-f(x) \le c.2^k - 4046$

$\Rightarrow 2c \in \mathbb{Z}$

Then $c \in z$ or $c = n+\frac{1}{2}$

If c is an integer

Then let $p(x) = f(x) - cx$

$p(x+a) = f(x+a) - cx - ac$

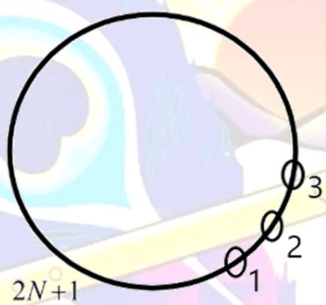
$\Rightarrow a \nmid p(x+a) - p(x)$

Then $c = n + \frac{1}{2}$ for some $n \in N$.

04.

Let $k \geq 1$ and $N > 1$ be two integers. On a circle are placed $2N+1$ coins all showing heads. Calvin and Hobbes play the following game. Calvin starts and on his move can turn any coin from heads to tails. Hobbes on his move can turn at most one coin that is next to the coin that Calvin turned just now from tails to heads. Calvin wins if at any moment there are k coins showing tails after Hobbes has made his move. Determine all values of k for which Calvin wins the game.

04. Solution



Case-1:

Let us assume that $k > N$, then by Pigeonhole Principle there exist at least two coins which are adjacent each other and which are the moves of Calvin and these adjacent coins are having tails on the up face. Then Hobbes always can turn. Another one among these two coins such that one will becomes head.

It at any moment there are k coins showing tais after Hobbes has made his move. So in this case Calvin never Wins.

Case-2:

Let $k \leq N$

So if $k \leq N$ then Calvin can moves alternatively by leaving one and another then he can reach always up to N element such that he always Wins.

Hence Calvin can get at most (N +1) tails to win.

05.

Euler marks n different points in the Euclidean plane. For each pair of marked points, Gauss writes down the number $\lfloor \log_2 d \rfloor$ where d is the distance between the two points. Prove that Gauss writes down less than $2n$ distinct values.

Note: For any $d > 0$, $\lfloor \log_2 d \rfloor$ is the unique integer k such that $2^k \le d < 2^{k+1}$.

05. Solution

$n = 2,3,4 \quad \Rightarrow {}^{n}c_2 < 2n$

$\Rightarrow$ true for $n = 2,3,4$

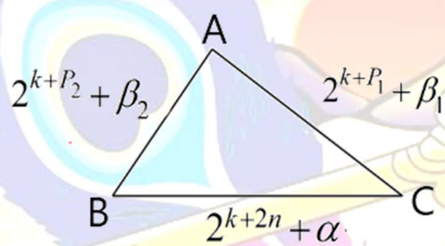
Let $n \ge 5 \Rightarrow {}^{n}c_2 \ge 2n$

Let Consider the set

$S = \{k+1,\ k+2, ..., k+2n\}$

Then we choose a triangle with one of side is largest i.e

$d_{\max} = 2^{k+2n} + \alpha, \quad \alpha \in \left[0, 2^{k+2n}\right)$



$i^{\beta t}\left[0, 2^{k+P_i}\right)$

$P_i \in \left[0, 2n\right)$

$i = 1,2$

Applying triangle inequality

$BC - AB < AC$

$2^{k+2n} + \alpha - \left(2^{k+P_2} + \beta_2\right) < 2^{k+P_1} + \beta_1$

$2^{k+2n} - 2^{k+P_2} < 2^{k+P_1} + \underbrace{\beta_1 - \alpha + \beta_2}$

Case-1:

If $\beta_1 + \beta_2 < \alpha$

$2^{k+P_1} + \beta_1 + \beta_2 + ... < 2^{k+P_1}$

$\Rightarrow 2^{k+2n} - 2^{k+P_2} < 2^{k+P_1}$

$\Rightarrow 2^{2n} - 2^{P_2} < 2^{P_1}$

$\Rightarrow 2^{2n} - 2^{P_2} < 2^{2n-k_1}$

$\Rightarrow 2^{2n+k_1} - 2^{P_2+k_1} < 2^{2n}$

$\Rightarrow 2^{2n}\left(2^{k_a} - 1\right) < 2^{P_2+k_1}$

$\Rightarrow 2^{2n} < \dfrac{\left(2^{P_2+k_1}\right)}{\left(2^{k_1-1}\right)}$ $\qquad 2^{k_1} > 2$

$\Rightarrow 2^{2n} < \dfrac{2^{P_2+k_1}}{\left(2^{k_1-1}\right)} \le \dfrac{2^{P_2+k_1}}{\left(2^{k_1-1}\right)}$

$\Rightarrow 2^{2n} < 2^{P_2+1}$

$\boxed{2n < P_2 + 1}$

$\because \; P_2 < 2n$

Which is contradiction to our solution

Case-2:

$\beta_1 + \beta_2 < \alpha$

$\beta_1 + \beta_2 - \alpha > 0$

Same Contradiction in left

General Case:

Consider any Random point from set of any Random set of $2n$ elements.

By using the previous argument

We can consider two cases:

Case1:

Then exist an element

$k + 2n + r_1$

$\therefore r_1 \ge 1$

Maximum

$2n \to 2n + r_1$

Same contradiction

06.

Euclid has a tool called *cyclos* which allows him to do the following:

- Given three non-collinear marked points, draw the circle passing through them.
- Given two marked points, draw the circle with them as endpoints of a diameter.
- Mark any intersection points of two drawn circles or mark a new point on a drawn circle.

Show that given two marked points, Euclid can draw a circle centered at one of them and passing through the other, using only the cyclos.

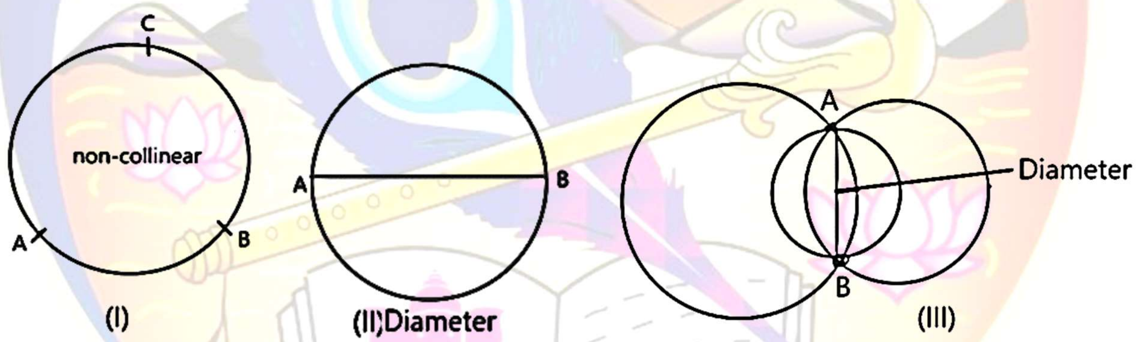
06. Solution

Cyclos:

I) Given three non-collinear marked points, draw the circle passing through them.

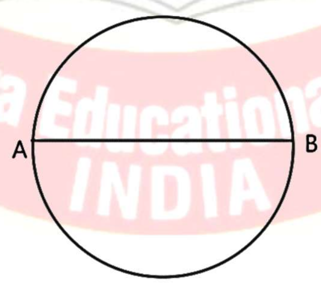
II) Given two marked points, draw the circle with them as end points of a diameter.

III) Mark any intersection points of two drawn circles or mark a new point on a drawn circle



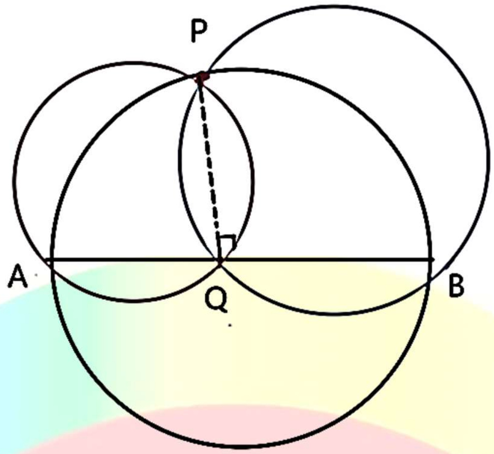
Now given two points let (A, B)

First Draw line AB and draw circle with diameter AB using Cyclos.



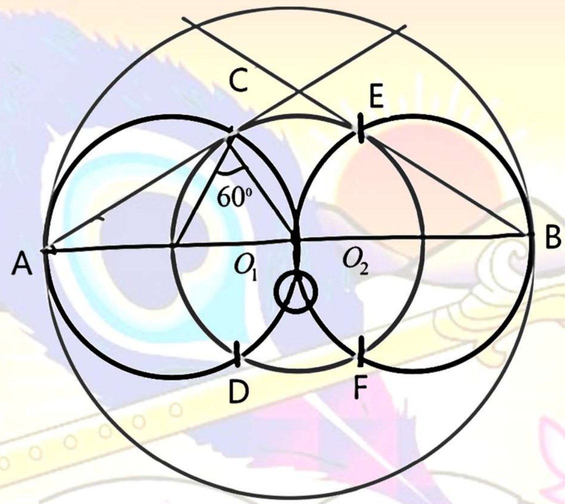
Now choose any Random point on the circle let P

Then draw circle using statement 2 from A and B



$PQ \perp AB$

So we always can final Mid-point for AB from above process



$\Rightarrow CO_1 = CO \Rightarrow COO_1$ is an equilateral

In the same way we can construct an equilateral triangle ABC^1

$\Rightarrow C^1A = C^1B = AB$

In the same way

ABB^1 is an equilateral triangle.

$\Rightarrow$ Circle with B, B^1, C^1 is a circle with AB as radius