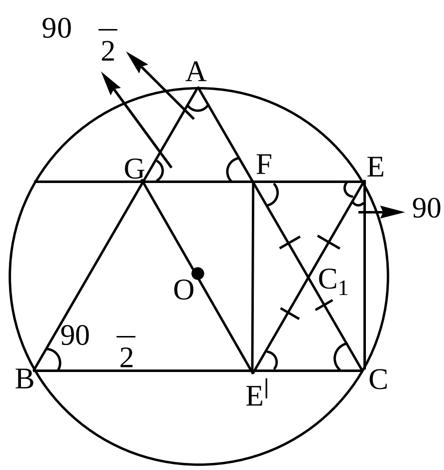
INMO (Indian National Mathematical Olympiad) - Solutions - 2024

1. In triangle ABC with CA = CB, point E lies on the circum circle of ABC such that ECB 90 . The line through E parallel to CB intersects CA in F and AB in G. Prove that the centre of the circum circle of triangle EGB lies on the circum circle of triangle ECF.

Solution:



Let C_1 be the circumcentre of ECF, then $C_1E \quad C_1F \quad C_1C$.

Let $E^{|}$ be the reflection of E w.r.t C_1.

We have AF = GF

OC is angular bisector of C (as ABC is isosceles)

$$OCE \quad 90 \quad \frac{}{2}$$

$$OEC \quad 90 \quad \frac{}{2}$$

$$COE$$

$$CAE \quad \frac{}{2}$$

AF = FE, AF = GF GF = FE

Now, GF = FE, $EC_1 \quad C_1E^{|}$ & $GEE^{|}$ is common

$FEC_1 \quad GEE^{|} \quad EE^{|} \quad E^{|}G$

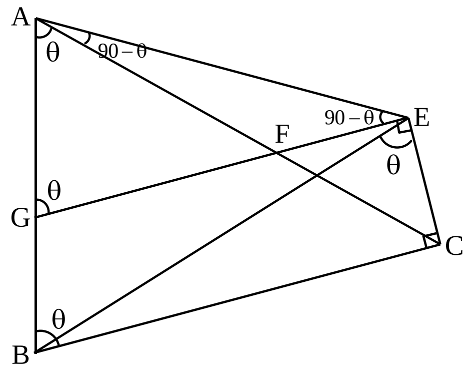
From the similarity, we have $EGE^{|} \qquad BGE^{|} \quad 90 \quad \frac{}{2}$

$BE^{|} \quad GE^{|} \quad EE^{|}$

This means $E^{|}$ is the circumcentre of BGE

But $E^{|}$ is also the reflection of E about C_1, and hence $E^{|}$ lies on the circumcircle of FEC.

Alternative solution :



Given that BC EC and GE || BC

BCE CEG 90

CA CB

BAC CBA

But as ABCE is cyclic, we have

BEC BAC & BAC BCE 180

BAC 90 180

BAC 90

FAE 90

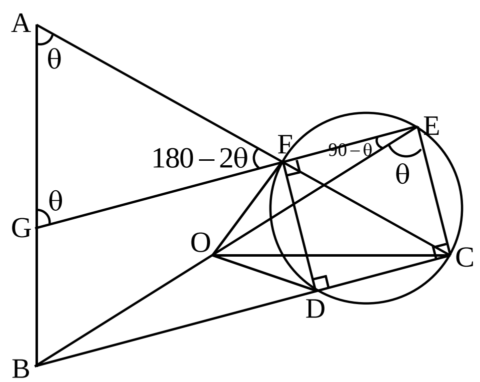
As GE || BC, AGE ABC

AEG 90

AFE 2 2 AGE

But as AGE is right angled and F lies o hypotenuse,

we have E is circumcentre of AGE GF FE



Draw FD GE D lies on the circle

Let O = mid point of BE

As BCE = 90° , BE is diameter and O is centre of ABC

BOC 2 BAC 2

COE 180 2

CFE AFG 180 2

But CFE COE 180 2

So, □COFE is cyclic and □DOEC is cyclic

DOE DCE 180

DOE 90

So FD is perpendicular bisector of GE

And DO is perpendicular bisector of BE

D is circumcentre of BGE lies on circumcentre of CFE

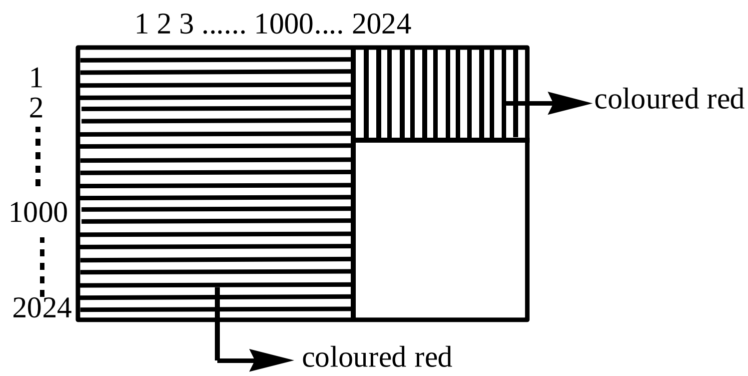
2. All the squares of a 2024 x 2024 board are coloured white. In one move Mohit can select one row or column whose every square is white, choose exactly 1000 squares in this row or column, and colour all of them red. Find the maximum number of squares that Mohit can colour red in a finite number of moves

Solution:

Mohit cannot select any row or column because it is clearly mentioned in the question that all the squares should be white. Without loss of generality assume Mohit first selected the rows. Then he should select 2024 rows. Now Mohit can select $\leq 2024-1000=1024$ columns (Mohit can select 1024 columns if all rows are identically coloured)

$\therefore$ Mohit can choose a maximum of 2024 + 1024 =3048 (including rows and columns)

Therefore Mohit selected $3048\times 100=3048000$ squares and coloured them. (Below is an example)



3. Let ‘p’ be an odd prime number and a, b, c be integers so that the integers

$a^{2023}+b^{2023}, b^{2024}+c^{2024}, c^{2025}+a^{2025}$

are all divisible by ‘p’. Prove that ‘p’ divides each of a, b and c.

Solution:

Let $p\times a$, then $p\times b$ and $p\times c$ (trivial)

From question,

$b^{2023}\equiv -a^{2023}\left(\text{mod } p\right)\rightarrow\left(1\right)$

$c^{2024}\equiv -b^{2024}\left(\text{mod } p\right)\rightarrow\left(2\right)$

$c^{2025}\equiv -a^{2025}\left(\text{mod } p\right)\rightarrow\left(3\right)$

Multiply (1) by ‘b’ and substitute in (2)

$c^{2024}\equiv a^{2023}b\left(\text{mod } p\right)$

Multiply by ‘c’ and substitute in (3)

$\boxed{a^{2}\equiv -bc\left(\text{mod } p\right)}$

From 1st equation,

$a\left(a^{2}\right)^{1011}\equiv -b^{2023}\left(\text{mod } p\right)$

$ac^{1011}\equiv b^{1012}\left(\text{mod } p\right)$ [As p won’t divide b]

$a^{2}c^{1011}\equiv ab^{1012}\left(\text{mod } p\right)$

$c^{1012} \equiv -ab^{1011} \pmod{p}$

$c^{2024} \equiv a^2 b^{2022}$

$-a^{2025} \equiv a^2 b^{2022} c$ [Using 3rd equation]

$-a^{2023} \equiv b^{2022} c$

$b^{2023} \equiv b^{2023} c$ [Using 2nd equation]

$\boxed{b \equiv c \pmod{p}}$

So, using 2nd equation,

$c^{2024} \equiv -c^{2024} \pmod{p}$

So, p/b and $p/c \Rightarrow p/a$

Contradiction

So, 'p' has to divide each of a, b, c.

4. A finite set S of positive integers is called cardinal if S contains the integer $|S|$, where $|S|$ denotes the number of distinct elements in S. Let f be a function from the set of positive integers to itself, such that for any cardinal set S, the set $f(S)$ is also cardinal. Here $f(S)$ denotes the set of all integers that can be expressed as $f(a)$ for some a in S. Find all possible values of $f(2024)$.

Note: As an example, $\{1,3,5\}$ is a cardinal set because it has exactly 3 distinct elements, and the set contains 3.

Solution: Considering the singleton cardinal set $\{1\}$. We see that $f(1)=1$. The cardinal set $\{1,2\}$ gets mapped to $\{1, f(2)\}$, so $f(2)$ must be 2 or 1.

Case 1. Suppose $f(2)=1$. Now $\{2,2024\}$ is a cardinal set, and therefore so is $\{1, f(2024)\}$.

This means $f(2024)$ is 1 or 2.

Case 2. Suppose $f(2)=2$.The cardinal set $f(\{1,2,3\})=\{1,2,f(3)\}$ shows that $f(3)\in\{1,2,3\}$, but the cardinal set $f(\{2,3\})=\{2,f(3)\}$ proves $f(3)$ cannot be 2. Thus there are two sub – cases.

Subcase (i). $f(3)=1$. Then the set $\{1,3,2024\}$ is cardinal, hence so is $\{1, f(2024)\}$, implying, as before, $f(2024)\in\{1,2\}$.

Subcase (ii). $f(3)=3$. In this case, we show via induction that $f(n)=n$ for all $n\in\mathbb{N}$.

The base cases $n=1,2,3$ are already known. Now consider $n\geq 4$, and assume $f(k)=k$ for all $k<n$. Consider the cardinal $f(\{1,2,......,n\})=\{1,2,...,n-1, f(n)\}$ which implies $f(n)\in\{1,2,....,n\}$ However, consider the $n-1$ element cardinal set $\{1,2,......,n\}\setminus\{n-2\}$. For its image to be cardinal $f(n)$ cannot equal any number in $\{1,2,...,n-1\}\setminus\{n-2\}$; else its cardinality would be $n-2$, which isn't in the set. So $f(n)\in\{n-2,n\}$.

Finally, consider the $n-2$ element set $\{1,2,....,n\}\setminus\{n-1,n-3\}$. If $f(n)=n-2$, its image would only have $n-3$ and the induction is complete. In particular, $f(2024)=2024$.

Thus the only possible values of $f(2024)$ are 1, 2 and 2024.

5. Let points $A_1, A_2, and\ A_3$ lie on the circle Γ in counter – clockwise order, and let P be a point in the same plane. For $i \in \{1,2,3\}$, let T_i denote the counter – clockwise rotation of the plane centred at A_i, where the angle of the rotation is equal to the angle at vertex A_i in $\Delta A_1A_2A_3$. Further, define P_i to be the point $T_{i+2}\left(T_i\left(T_i+1(P)\right)\right)$, where indices are taken modulo 3 $(i.e., T_4 = T_1\ and\ T_5 = T_2)$.

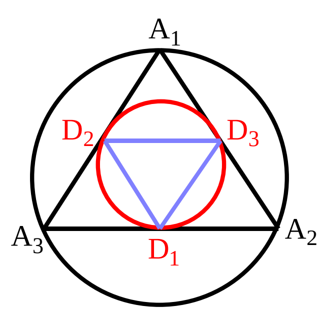
Prove that the radius of the circumcircle of $\Delta P_1P_2P_3$ is at most the radius of Γ.

Solution: Fix an index $i \in \{1,2,3\}$. Let D_1, D_2, D_3 be the points of tangency of the incircle of triangle $\Delta A_1A_2A_3$ with its sides A_2A_3, A_3A_1, A_1A_2 respectively.

The key observation is that given a line ℓ in the plane, the image of ℓ under the mapping $T_{i+2}\left(T_i\left(T_{i+1}(\ell)\right)\right)$ is a line parallel to ℓ. Indeed, ℓ is rotated thrice by angles equal to the angles of $\Delta A_1A_2A_3$, and the composition of these rotations induces a half – turn and translation on ℓ as the angles of $\Delta A_1A_2A_3$ add to 180°. Since D_i is a fixed point of this transformation (by the chain of maps $D_i \xrightarrow{T_{i+1}} D_{i+2} \xrightarrow{T_i} D_{i+1} \xrightarrow{T_{i+2}} D_i$), we conclude that the line $\overline{PD_i}$ maps to the line $\overline{P_iD_i}$. But the two lines are parallel and both of them pass through D_i hence they must coincide, so D_i lies on $\overline{PP_i}$. Further, each rotation preserves distances, hence P_i is the reflection of P in D_i.

In other words, the triangle $P_1P_2P_3$ is obtained by applying a homothety with ratio 2 and centre P to the triangle $D_1D_2D_3$. Thus, the radius of the circumcircle of $\Delta P_1P_2P_3$ is twice the radius of the circumcircle of $\Delta D_1D_2D_3$. i.e., twice the radius of the incircle of $\Delta A_1A_2A_3$, which is known to be at most the radius of the circumcircle Γ.

Remark. The conclusion used the fact that in a triangle ABC with incentre I and in radius r, and circumcentre O and circumradius R, we have the inequality $R \geq 2r$. This is called Euler's Inequality. The standard proof is that $0 \leq OI^2 = R^2 - Pow\left(I,(O,R)\right) = R^2 - 2Rr$. The last equality holds as Pow (I, (O, R)) = IA. IM where M is the midpoint of minor arc $\overset{\frown}{BC}$ in the circumcircle of ABC, and because $IA = \dfrac{r}{\sin\dfrac{A}{2}}$ $and\ IM = MB = \dfrac{a}{2\cos\dfrac{A}{2}} = \dfrac{2R\sin A}{2\cos\dfrac{A}{2}} = 2R\sin\dfrac{A}{2}$ by using "the trident lemma" and the double – angle sine formulas.



6. For each positive integer $n \ge 3$, define A_n and B_n as

$$A_n = \sqrt{n^2+1} + \sqrt{n^2+3} + \cdots + \sqrt{n^2+2n-1},$$

$$B_n = \sqrt{n^2+2} + \sqrt{n^2+4} + \cdots + \sqrt{n^2+2n}$$

Determine all positive integers $n \ge 3$ for which $[A_n] = [B_n]$.

Note: For any real number x, $[x]$ denotes the largest integer N such that $N \le x$.

Solution:

Let $M = n^2 + \frac{1}{2}n$

Case (i):

$$(B_n - A_n) = \sum_{k=1}^{n}\left(\sqrt{n^2+2k} - \sqrt{n^2+2k-1}\right) = \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+2k}+\sqrt{n^2+2k-1}} < \sum_{k=1}^{n}\frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

Case (ii):

$$(A_n - n^2) = \sum_{k=1}^{n}\left(\sqrt{n^2+2k-1} - n\right) = \sum_{k=1}^{n}\frac{2k-1}{\sqrt{n^2+2k-1}+n} < \sum_{k=1}^{n}\frac{2k-1}{n+n} = \frac{n^2}{2n} = \frac{n}{2}$$

as $\sum_{k=1}^{n}(2k-1) = n^2$, proving $A_n - n^2 < \frac{n}{2}$ or $A_n < M$

Similarly,

$$(B_n - n^2) = \sum_{k=1}^{n}\left(\sqrt{n^2+2k} - n\right) = \sum_{k=1}^{n}\frac{2k}{\sqrt{n^2+2k}+n} > \sum_{k=1}^{n}\frac{2k}{(n+1)+n} = \frac{n(n+1)}{2n+1} > \frac{n}{2}$$

as $\sum_{k=1}^{n}(2k) = n(n+1)$, so $B_n - n^2 > \frac{n}{2}$ hence $B_n > M$

By Case (ii), we see that A_n and B_n are positive real numbers containing M between them. When 'n' is even, M is an integer. This implies $[A_n] < M$, but $[B_n] \ge M$, which means we cannot have $[A_n] = [B_n]$.

When 'n' is odd, M is a half-integer, and thus $M - \frac{1}{2}$ and $M + \frac{1}{2}$ are consecutive integers.

So the above two cases imply

$$M - \frac{1}{2} < B_n - (B_n - A_n) = A_n < B_n = A_n + (B_n - A_n) < M + \frac{1}{2}$$

This shows $[A_n] = [B_n] = M - \frac{1}{2}$.

Thus, the only integers $n \ge 3$ that satisfy the conditions are the odd numbers and all of them work.